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Quiz!

§3.4 Coordinates

From last time:

if $\mathcal{B}$ is given, then coordinates are uniquely determined. In particular,

if $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ and $\vec{v} \in V$, and $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

then $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$. And vice versa.

Th 3.4.2 we'll use this shortly

Let $\mathcal{B}$ be a basis for a v.s. V (in $\mathbb{R}^n$).

Let $\vec{x}, \vec{y} \in V$. Then we have linearity:

(a) $[\vec{x} + \vec{y}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$

(b) $[k\vec{x}]_{\mathcal{B}} = k[\vec{x}]_{\mathcal{B}}$ $\forall k \in \mathbb{R}$ and $\forall \vec{x} \in V$.

We also started talking about change of basis matrices (we called it S last time) but we'll come back to that.

Where is this going?

So far we've talked about vectors in terms of the standard basis, and we've talked about lin. transf. in terms of the standard basis (for each vector space).

Now suppose we have

- $T: \mathbb{R}^n \to \mathbb{R}^n$ defined by a (square) matrix A all in terms of the std basis for $\mathbb{R}^n$. (as we've been doing so far)
- A basis $\mathcal{B}$ for $\mathbb{R}^n$, say $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$.

Our job: • express T in terms of $\mathcal{B}$

• find a dictionary to go between the 2 expressions

Idea:

When we were doing everything with the standard basis $\mathcal{S} = \{\vec{e}_1, \dots, \vec{e}_n\}$ we said if $T: \mathbb{R}^n \to \mathbb{R}^n$ is defined by a matrix A then for any vector $\vec{x}$ we have

$$\vec{x} = [\vec{x}]_{\mathcal{S}}$$

(e.g. $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\vec{e}_1 + 2\vec{e}_2$ so $[\vec{x}]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.)

$$\Rightarrow \quad A = \left[T(\vec{e}_1) \mid \cdots \mid T(\vec{e}_n) \right]$$

$$= \left[[T(\vec{e}_1)]_{\mathcal{S}} \mid \cdots \mid [T(\vec{e}_n)]_{\mathcal{S}} \right]$$

this is just a rewriting of what we're been doing

$\Rightarrow$ we said since $T(\vec{x}) = A\vec{x}$, we have

$$(*) \quad [T(\vec{x})]_{\mathcal{S}} = A[\vec{x}]_{\mathcal{S}}$$

(says the same thing in this case).

Of course at the time there was no reason to complicate the notation like this, but now we'll do it so the next result seems more reasonable.

Now we want to replace $\mathcal{S} = \{\vec{e}_1, \ldots, \vec{e}_n\}$ by $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$.

<u>Thm</u> (3.4.3) (and def'n)

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a lin transf.

Let $\mathcal{B}$ be a basis for $\mathbb{R}^n$.

Then there exists a unique $n \times n$ matrix B so that

$$\left.\begin{array}{l} [T(\vec{x})]_{\mathcal{B}} = B[\vec{x}]_{\mathcal{B}} \\ \text{for all } x \in \mathbb{R}^n. \end{array}\right\}$$ compare with (*)

B is called the $\boxed{\mathcal{B}\text{-matrix of } T.}$ Specifically:

$$B = \Big[[T(\vec{v}_1)]_{\mathcal{B}} \;\Big|\; \cdots \;\Big|\; [T(\vec{v}_n)]_{\mathcal{B}} \Big]$$

This completes the 1st part of our job: express T in terms of $\mathcal{B}$

So in particular we replace A by B.

Do an example to illustrate the theorem, then prove the theorem.

<u>Ex</u> $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ in terms of $\vec{e}_1, \vec{e}_2$

Say $\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\vec{v}_2} \right\}$ and $\vec{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$. What does Th 3.4.3 say here?

Using standard basis:

$$T(\vec{v}_1) = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} = 8\overbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}^{\vec{v}_1} - 2\overbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}^{\vec{v}_2} \Rightarrow [T(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$$T(\vec{v}_2) = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} = 11\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow T(\vec{v}_2) = \begin{bmatrix} 11 \\ -3 \end{bmatrix}$$

So $\quad B = \begin{bmatrix} 8 & 11 \\ -2 & -3 \end{bmatrix}$

Now

$$\vec{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} = 6\vec{v}_1 - \vec{v}_2 \qquad \text{so} \quad [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 19 \\ 28 \end{bmatrix}$$

$$= 37 \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{v}_1} - 9 \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\vec{v}_2} \implies [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 37 \\ -9 \end{bmatrix}$$

Check:

The theorem says

$$[T(\vec{x})]_{\mathcal{B}} = B[\vec{x}]_{\mathcal{B}}$$

$$\begin{bmatrix} 37 \\ -9 \end{bmatrix} = \begin{bmatrix} 8 & 11 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 6 \\ -1 \end{bmatrix} \qquad \checkmark$$

Proof of Th 3.4.3

the assertion is

for all $x \in \mathbb{R}^n$, we have $[T(\vec{x})]_{\mathcal{B}} = B[\vec{x}]_{\mathcal{B}}$ where

$$B = \Big[[T(\vec{v}_1)]_{\mathcal{B}} \,\Big|\, \cdots \,\Big|\, [T(\vec{v}_n)]_{\mathcal{B}} \Big].$$

Say $\vec{x} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$ so $[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

$$[T(\vec{x})]_{\mathcal{B}} = [T(c_1\vec{v}_1 + \dots + c_n\vec{v}_n)]_{\mathcal{B}}$$

$$= [c_1 T(\vec{v}_1) + \dots + c_n T(\vec{v}_n)]_{\mathcal{B}} \quad \text{since } T \text{ is linear}$$

$$= c_1[T(\vec{v}_1)]_{\mathcal{B}} + \dots + c_n[T(\vec{v}_n)]_{\mathcal{B}} \quad \text{Th } 3.4.2$$

$$= \underbrace{\begin{bmatrix} [T(\vec{v}_1)]_{\mathcal{B}} & \Big| & \cdots & \Big| & [T(\vec{v}_n)]_{\mathcal{B}} \end{bmatrix}}_{B} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$= B[\vec{x}]_{\mathcal{B}}$$

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